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Constructing Some Embedded of Petersen Subfamilies in the Klein Bottle Using Link Diagram

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Abstract. We discovered the embedding of K_6 in the Klein bottle is differed by the use of the link diagram, where the number of the components of the link diagram are not always equal. This result helps to construct some subfamilies of the embedding of Petersen family in the Klein bottle.

Keywords: embedded graph, components of link, Klein bottle, Petersen family.

1- Introduction

This paper has introduced the relationship between the graph theory and knot theory. As we know, graph theory is a part of combinatorics, where knot theory is a part of geometric topology.

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The working depends on the embedded graph which means each face of the embedded graph is homeomorphic to a disc; so is called it a cellularly embedded graph [is a graph drawn on a surface in such a way that edges only intersect at their ends. (Adams, 2004)

Let $G = (V, E)$ be a connected graph , $v \in V$ is a vertex of degree 3, $H = (W, F)$ is obtained from G by deleting v and add edges between each pair of these vertices pass from v as in the fig. 1 below:

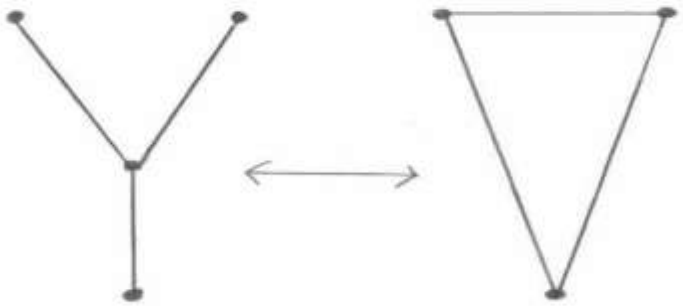

Fig. 1 $Y \leftrightarrow \Delta$ exchange.

This operation is called Y $\rightarrow \Delta$ exchange, the converce of this deal is called $\Delta \rightarrow$ Y exchange (Robertson, 1993). Petersen family contains seven graphs. These graphs come from a complete graph K6 by using Y $\rightarrow \Delta$ exchange or $\Delta \rightarrow$ Y exchange (Mphako, 2002).

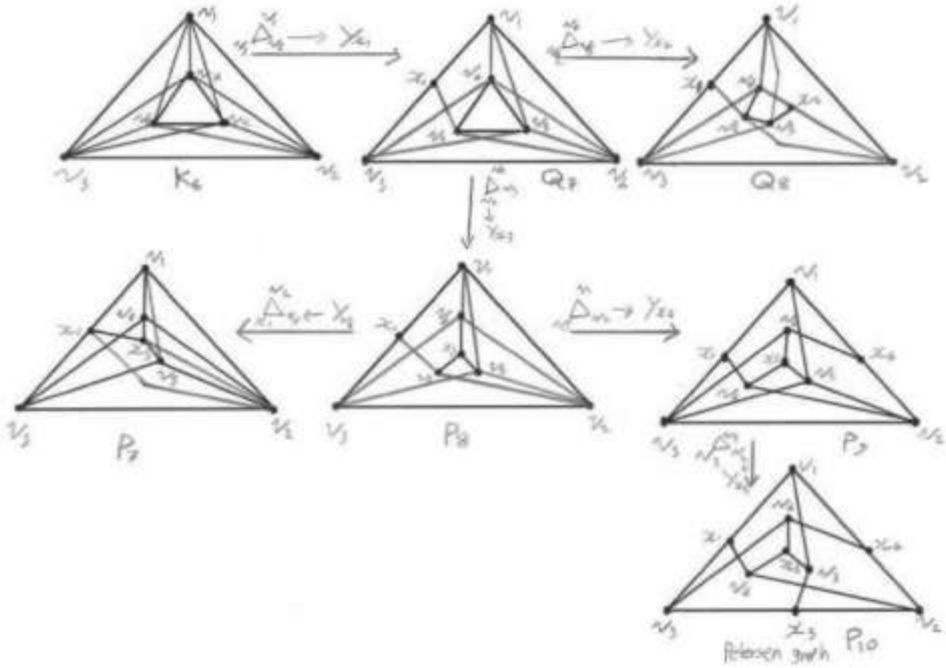

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Fig. 2 The abstract Petersen family as it appeared in (Tawfik, 2013). Each arrow refers to $\Delta \leftrightarrow Y$ exchanges.

The Q7 graph, the second graph in this family is got by choosing any triangle of K6. Q7 contains triangles when we exchange one of these triangles to Y, we get Q8 or P8. The different between Q8 and P8 that is Q8 does not have any triangle. P8 having one or two triangle do not share an edge, so we can obtain P8 by exchanging the triangle in P8 to Y, this operation gives P10. P10 is a regular graph-3 contains ten vertices, it is called Petersen graph(Holten, 1983). One element in Petersen family is P7, which we take it from P8 by exchanging the new Y appears in P8 to Δ.

A link is an embedding of any disjoint circles in R^3, and each circle is called a component of a link (Tawfik, 2013).

Our research obtains the diagrams of unoriented links in R^3 and the corresponding graphs of Petersen family embedded on the Klein bottle.

A diagram D of a link L is essentially a drawing of it on a surface (Ellis-Monaghan, 2010). It is drawing on S^2 for links in S^3.

We suppose $\mu(D(G))$ the component of the link diagram`s number of the embedding graph G (Mphako, 2002).

This article focuses on one relationship between the graph and the link diagram. This relationship through replaces each edge of the graph to the crossing of the link diagram, as shown in fig. 3.

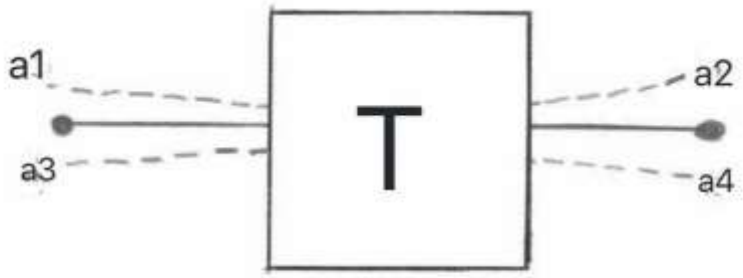

Fig. 3 An edge is changed to a tangle. The dotted lines which are marked a1, a2, a3, and a4 are strands of the link diagram, and the edge appears as the straight line joining two vertices. It is replaced by the tangle. The strand a1 is joined to a4 and a3 is joined to a2.

The history of this relationship comes from several previous studies. The number of component on the embedded graph has been studied in (Martin, 1978) that

$T(G;-1,-1) = (-1)^{q(G)} (-2)^{\mu(D(G))-1}$ − − − − − − − − − ∗

Where $\mathcal{T}(G;x,y)$ is the polynomial and $q(G)$ is the number of edges in G, while this polynomial is shown in (Martin, 1978) by generalizing it in the projective plane and in the torus.

$T(G;-1,-1)$ is calculated in (Pisanaki, 2004) for 2-sums, wheels and fans of graphs. (Pisanaki, 2004) has described this number of μ as the number of straight ahead walks in medial graphs. Also in (Huggett,2015) has discussed the largest number of $\mu(D(G))$ for the plane graphs. (Huggett, 2015) has studied embedded graphs on the surfaces of positive genus whose links have the largest possible number of components. Section2 discuses some relation result introduced in (Tawfik, 2013).

In this paper we work with the embedding of Petersen family which is the interesting family of the embedded graphs. All the graphs in this family can be cellularly embedded in the tours as an orientation surface and the Klein bottle as a non-orientation surface. As we Know that the embedded $\Delta\leftrightarrow Y$ exchanges do not change the value of μ by using the third Reidemeister move (Tawfif, 2013) is defined in (Cromwell, 2004). This fact means the value of μ does not alter or constant for each embedding of Petersen family.
The cellularly embedding of Petersen family are not unique. That means we have different values of μ depending on the embedding. So that we obtain the subfamilies of Petersen from the embedded $\Delta\leftrightarrow Y$ exchanges.

2-Basic Results:

In this section, we introduce the embedded $\Delta\leftrightarrow Y$ exchanges for the Petersen family in the tours. However these results discussed in (Tawfik, 2013).
This article has been built by depending on the following basic results:

Theorem 1. (Tawfik, 2013)

Let G and H be embedded graphs, where H is got from G using embedded $\Delta\leftrightarrow Y$ exchanges. Then G and H have the same number of components of a link diagram.
This theorem performs to study the embedding of K6 in the tours and the Klein bottle.
The value of μ does not alter under embedded $\Delta\leftrightarrow Y$ exchanges, which means that embedded of Petersen family has the same number of components of μ. But the restriction of the embedded of $\Delta\leftrightarrow Y$ exchanges, do not get the same Petersen families.
The following theorem gives three subfamilies of Petersen from the embedded $\Delta\leftrightarrow Y$ exchanges, where $\mu = 3$.

Theorem 2. (Tawfik, 2013)

If the embedding of K6 in the tours shown in fig. 4 is taken, then the embedded $\Delta\leftrightarrow Y$ exchanges give the following three Petersen subfamilies:
a) {K6, Q7, Q8, P8, P7, P9, P10}.

b) {K6, Q7, P8, P7, P9, P10}.

c) {K6, Q7, P8, P7, P9}.

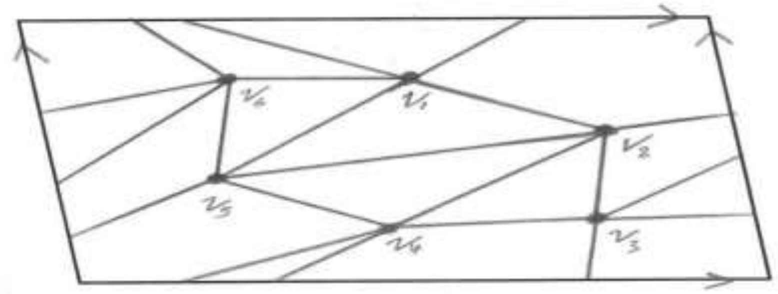

Fig. 4 The embedding of K6 in the tours of $\mu = 3$.

Fig.5 for another embedding of K6 which gives a different group of Petersen subfamilies having $\mu = 5$.

Theorem 3. (Tawfik, 2013)

If the embedding of K6 in the tours appeared in the above fig. 5 is given, then the embedded $\Delta \leftrightarrow$ Y exchanges award the following Petersen subfamilies:
a) {K6, Q7, Q8, P8, P7, P9}.

b) {K6, Q7, P8, P7, P9}.

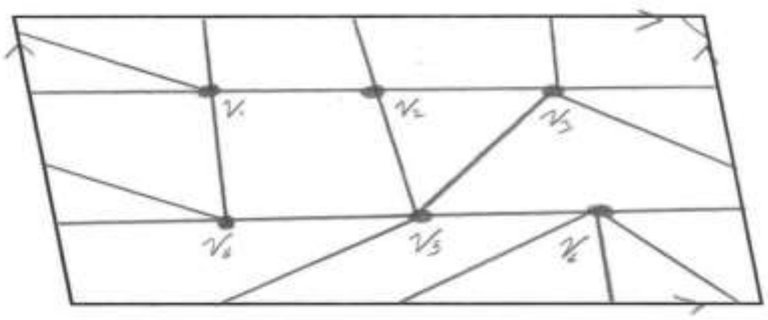

Fig. 5 The embedding of K6 in the tours having $\mu = 5$.

In the next embedding of K6 in the tours, $\mu(D(G)) = 7$ in fig. (6).

Theorem 4. (Tawfik, 2013)

If the embedding of K6 given in fig. 6 , then the embedded $\Delta \leftrightarrow$ Y exchanges gives raise the subfamily {K6, Q7, Q8, P8, P7, P9}.

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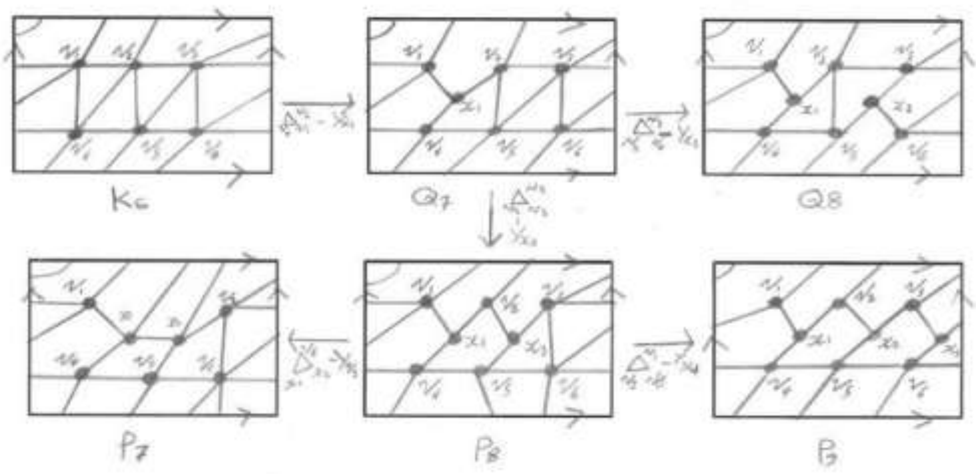

Fig. 6 The embedding of Petersen subfamily in the tours of $\mu = 7$.

3-Embedded of Petersen families in the Klein bottle:

Theorem 5. If the embedding of K6 in fig. 7 is taken, then the embedded $\Delta \leftrightarrow Y$ exchanges on this embedding give the following Petersen subfamilies:

a) { K6, Q7, Q8, P8, P7, P9 }.

b) { K6, Q7, P8, P7, P9, P10 }.

c) { K6, Q7, P8, P7, P9 }.

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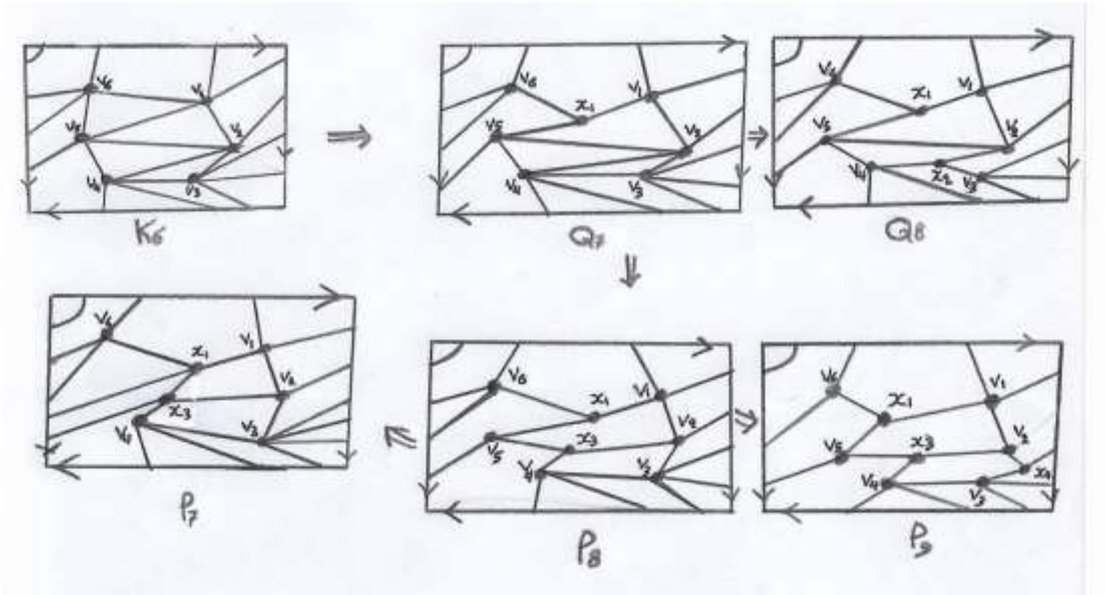

Fig. 7 The embedding of the first Petersen subfamily in the Klein bottole of $\mu = 3$.

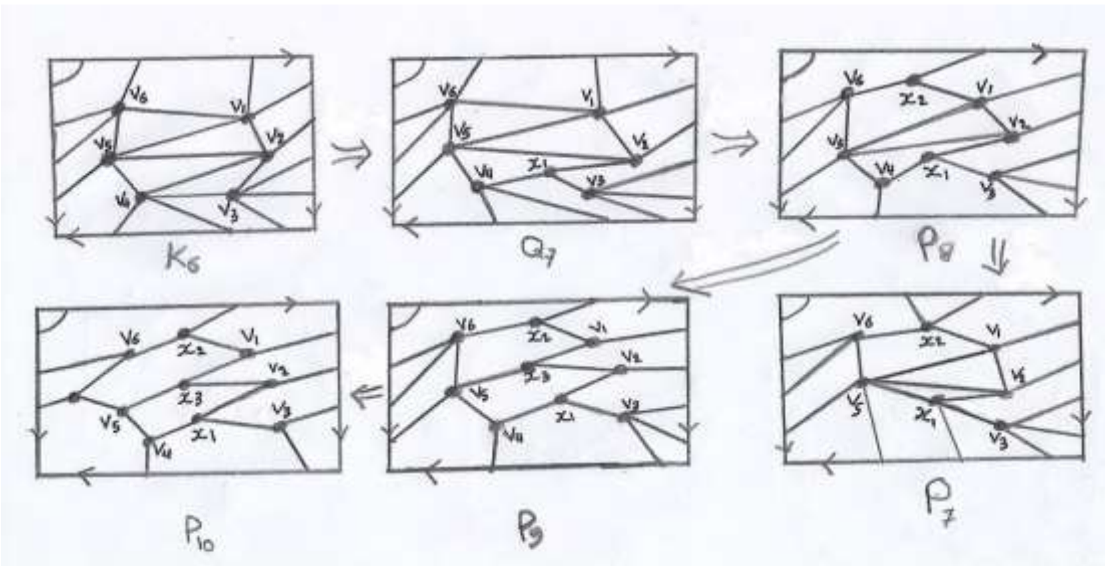

Fig. 8 The embedding of the second Petersen subfamily in the Klein bottole having $\mu = 3$, by using the embedding of K6 in fig. 7.

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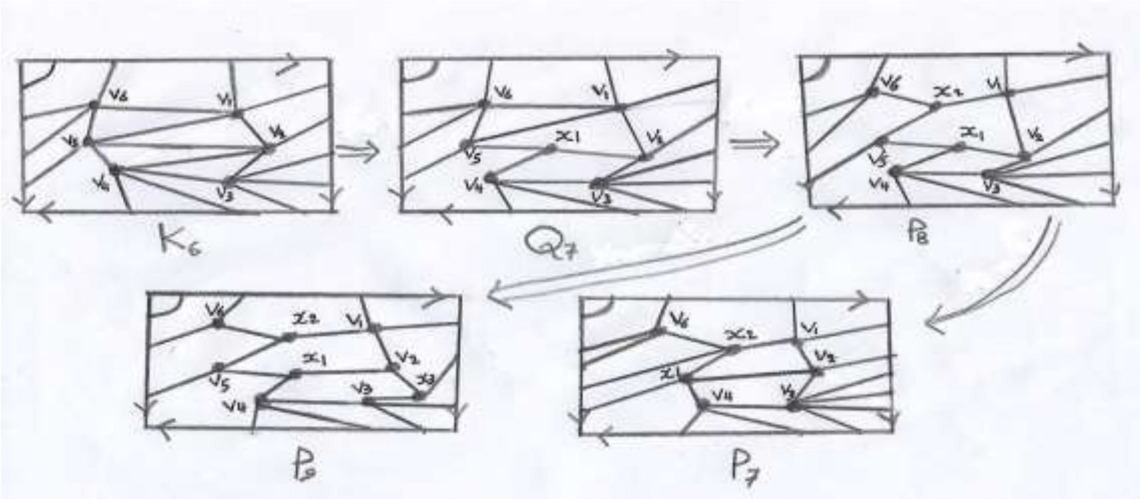

Fig. 9 The embedding of the third Petersen subfamily in the Klein bottole of $\mu = 3$, by using the embedding of K6 in fig. 7.

Proof:

a) In fig. 7, the embedding of K6 in Klein bottle contains seven triangles, every triangle is bounding a disc. This case having a pair of disjoint triangles $\{ v1, v5, v6 \}$ and $\{v2, v3, v4 \}$. If the triangle $\{ v1, v5, v6 \}$ is changed to Y_{x1} as in fig. 7, the embedding of Q7 is gotten with the new vertex $(x1)$.
Now the embedding of Q8 is gotten by changing the triangle $\{v2, v3, v4 \}$ to Y_{x2} shown in fig. 7, to be three elements of the new subfamily of Petersen.
There is a triangle $\{v2, v4, v5 \}$ in Q7 fig. 7, if it is changed to Y_{x3}, the embedding of P8 is got as in fig. 7. In the embedding of P8, there are three vertices $\{ x1, v5, x2 \}$ of degree three. These vertices give Y_{v5}, when it is change to the triangle $\{ x1, x3, v3 \}$ gets P7 in fig. 7. In the embedding of P8, there is a triangle $\{v2, v3, v6 \}$ the changing of this triangle to Y_{x4} gives us the embedding of P8 shown in fig. 7. The embedding of P8 does not contain any triangle and that means we have the first subfamily of Petersen embedding in Klein bottle contains six elements { K6, Q7, Q8, P8, P7, P9 }.
b) Return to the embedding of K6 in fig. 7 to change the triangle $\{v2, v3, v4 \}$ to Y_{x1}, this changing gives the embedding of Q7 shown in fig. 8. In the embedding of Q7 there is a triangle $\{ v1, v6, v4 \}$, if it is changed to Y_{x2}, we get the embedding of P8 as in fig. 8.
We use the same discussion in part (a) of this proof to obtain the embedding of P7 shown in fig. 8.
In the embedding of P8 fig. 8, there are two triangles, $\{ v1, v2, v5 \}$and $\{ v3, v5, v6 \}$. If one of these triangles changes to Y_{x3}, the embedding of P8 is obtained as In fig. 8. This embedding has one triangle, if it is changed to Y_{x4}, we get the embedding of P10 shown in fig. 8.
Now we have the second subfamily of Petersen in Klein bottle which it contains six elements { K6, Q7, P8, P7, P9, P10 }.

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The same subfamily can be got when we choose the triangle $\{v1, v2, v5\}$ in the embedding of K6 in fig. 7, and then choose any triangle in the embedding of Q7 except a triangle $\{v3, v2, v6\}$.
Also the same subfamily can be obtained when we choose a triangle $\{v3, v5, v6\}$ in the embedding of K6 in fig. 7, and then choose any triangle in the embedding of Q7 except a triangle $\{v2, v4, v5\}$.
The same discussion can give the equal embedding of this subfamily by choosing the triangle $\{v6, v5, v3\}$ and then one of the triangles $\{v2, v3, v4\}$ or $\{v1, v2, v5\}$.
c) Now let us return to the embedding of K6 in fig. 7, choose the triangle $\{v2, v4, v5\}$ and change it to Y_{x1}, to get the embedding of Q7 shown in fig. 9. This embedding contains four triangles, when we choose any triangle of them to change it to Y_{x2}, we get the embedding of P8 having one triangle or two triangles sharing in one edge. So the choosing one of appearing triangle gives the embedding of P9. Fig. 9 has chosen triangle $\{v1, v6, v5\}$.
To obtain P10 in fig. 9, we need a triangle $\{v1, v4, v3\}$ which is impossible because it does not bound a disc. That means we have the third Petersen subfamily of this embedding of K6. It contains five elements { K6, Q7, P8, P7, P9 }.

The same subfamily can be got when we choose a triangle $\{v2, v3, v6\}$, $\{v1, v6, v4\}$ or $\{v6, v5, v3\}$ in the embedding of Q7 in fig. 9. □

Theorem 6. If the embedding of K6 in fig. 10 is taken, then the embedded $\Delta \leftrightarrow Y$ exchanges on this embedding gives the following Petersen subfamily { K6, Q7, Q8, P8, P7, P9 }.

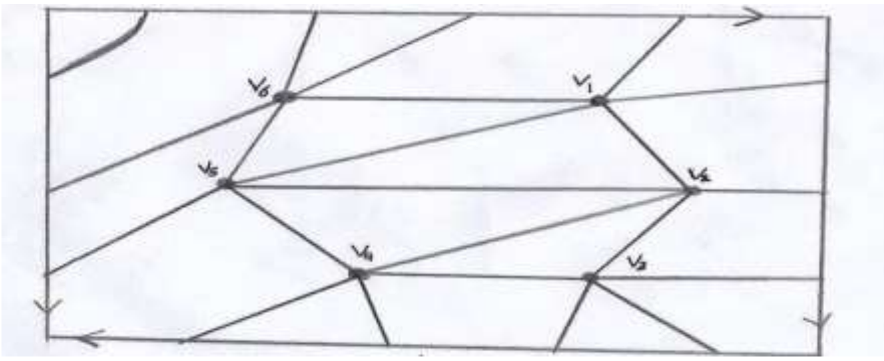

Fig. 10 The embedding of K6 in Klein bottle of $\mu = 5$.

Proof:

This embedding of K6 has six triangles bounding a disc and has three pairs of disjoint triangles. When we choose any triangle of any pair in the embedding of K6, we obtain the Petersen subfamily in the statement of this theorem. □

The embedding of K6 appears in fig. 10 gives the similar subfamily. The construct of this family has explained in the following theorem.

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Theorem 7. If the embedding of K6 in fig. 11 is taken then we can get the following Petersen subfamilies:

a) { K6, Q7, P8, P7 }.

b) { K6, Q7, P8, P7, P9 }.

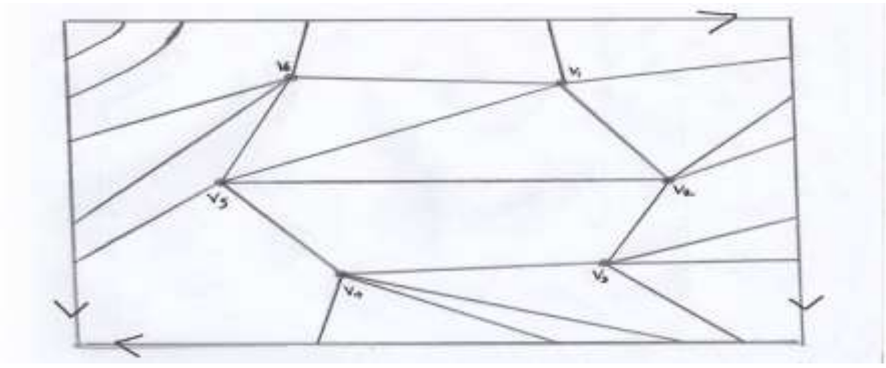

Fig. 11 The embedding of K6 in Klein bottle. This embedding gives two subfamilies in theorem 7.

Proof:

a) this embedding of K6 has six triangles each bounding a disc, but does not have a pair of disjoint triangles. That means the subfamilies is got from this embedding of K6 does not have the embedding of Q8. Let us choose the triangle $\{v1, v5, v6\}$ to change it to Y_{x1}, to get the embedding of Q7. Since each edge of this triangle is sharing with another triangle, then Q7 has two triangles which are sharing in one edge and the changing one of them give us the new subfamily contains just four elements { K6, Q7, P8, P7 }.

b) Return to the embedding of K6 in fig. 11, there are four triangles each sharing in two edges with other triangles in this embedding. These triangles are $\{v1, v6, v4\}$, $\{v6, v2, v4\}$, $\{v6, v2, v3\}$, $\{v6, v5, v3\}$.

Now let us change the triangle $\{v1, v6, v4\}$ to Y_{x1}, we get the embedding of Q7. This embedding has three triangles, when we choose any one , we get the embedding of P8 having one triangle bounding a disc or two triangles sharing in one edge.

That means we can get P9, but we cannot obtain P10 in this case of this embedding { K6, Q7, P8, P7, P9 }.

We can use the same way to find the same subfamily when we change the triangle $\{v6, v5, v3\}$ or change the triangle $\{v6, v2, v3\}$.

In this embedding of K6, we have one triangle $\{v1, v2, v5\}$ sharing of one edge with another triangle. All probabilities from changing the triangle $\{v1, v2, v5\}$ give the same family in (b) too. □

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