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The 1st International Conference on Sciences and Arts (ICMSA 2017)

المؤتمر الدولي الاول للعلوم والاداب

3 مايو 2017 - اربيل – العراق

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On Goldie* -T- lifting Modules

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Abstract. In this paper , we introduce β_T^* relation on the Lattic of submodules of a module M. We say that submodule X ,Y of M are β_T^* equivalent , X β_T^* Y, if $\frac{X+Y}{X} \leq \frac{X+H}{X}$ and $\frac{X+Y}{Y} \leq \frac{Y+H}{Y}$,for some H≪TM .We show that β_T^* relation is an equivalence relation and has good behavior with respect to addition of submodules and homomorphisms. This relation is used to define and study the class of Goldie*-T-lifting modules.

Keyword: T-small submodule, Goldie*-lifting module, H–supplemented, T-lifting module, Goldie*-T-lifting modules, T-H–supplemented.

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T-*حول مقاسات الرفع من النمط غولدي

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الخلاصة:

في هذا البحث , قدمنا مفهوم العلاقة β_T^* على مجموعة المقاسات الجزئية من مقاس M . يقال ان المقاسين الجزئيين X , Y من M يرتبطان بالعلاقة β_T^* , (X β_T^* Y) إذا وفقط اذا $\frac{X+Y}{X} \le \frac{H+X}{X}$ و $\frac{X+Y}{Y} \le \frac{H+Y}{Y}$, لبعض المقاس الجزئي الصغير H بالنسبة الى المقاس الجزئي T. برهنا ان العلاقة β_T^* هي علاقة تكافؤ ولها سلوك جيد بالنسبة للمقاسات الجزئية والتشاكلات .هذه العلاقة أستخدمت لتعريف ودراسة مقاسات الرفع من النمط غولدي*-T.

1-Introduction

Throughout this paper , rings are associative with identity and modules are unital left R-modules . Recall that a submodule N of an R-module M is small, denoted by $N \ll M$,if for any submodule X of M , N + X = M implies that X = M [1] .In [2], the authors introduced the concept of small submodule with respect to an arbitrary submodule .Let T be an arbitrary submodule of a module M .Recall that a submodule N of M is called T-small in M , denoted by $N \ll_T M$, in case for any submodule X of M , $T \subseteq N+X$ implies that $T \subseteq X$ .The notions of smallness and T-smallness coincide if T=M .If T=0 ,then every submodule of M is T-small in M .Also if $T \neq 0$,then $N \ll_T M$ implies that $T \nsubseteq N$ .Recall that a module M is H- supplemented if for every submodule A there is a direct summand D of M such that A+X = M if and only if D+X = M ,for every submodule X of M [3, P.95] .Recall that a module M is lifting if for every submodule A of M ,there a decomposition $M = D \oplus D`$ such that $D \subseteq A$ and $A \cap D` \ll M$ [4] .Recall that submodules X , Y of M are equivalent , X β^* Y if and only if $\frac{X+Y}{X} \ll \frac{M}{X}$ and $\frac{X+Y}{Y} \ll \frac{M}{Y}$ [5] .Recall that a module M is Goldie*- lifting, G*- lifting, if and only if for each submodule X of M there exists a direct summand D of M such that X β^*D [5].

These observations lead us to introduce the β_T^* relation.
Let T be submodule of M, We say submodules X ,Y of M are β_T^* equivalent, X β_T^* Y if and only if $\frac{X+Y}{X} \le \frac{X+H}{X}$ and $\frac{X+Y}{Y} \le \frac{Y+H}{Y}$, for some $H \ll_T M$.

In section 2 , we define and study β_T^* relation on the set of submodules of a module M .

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In section 3, we introduced the concepts of Goldie*-T- lifting , T- lifting and T- H -supplemented modules and we give basic properties and various characterizations for them .

Let R be a ring and M a left R–module .If X ⊆ M, then X ≤ M, X ≪ M, X≪TM , X ≤⨁M, and End (M) denote X is a submodule of M , X is a small submodule of M , X is T-small submodule of M , X is a direct summand of M and the ring of endomorphisms of M , respectively .

Recall that a submodule N of M is a projection invariant if e (N) ⊆ N , for each e = e2 ∈ End (M) . Other terminology and notation can be found in [3,4,7]

2- The β_T^* Relation .

In this section, we develop the basic properties of the β_T^* relation .

Definition 2.1 Let T be a submodule of a module M .We say submodules X ,Y of M are β_T^* equivalent , X β_T^* Y ,if $\frac{X+Y}{X} \leq \frac{X+H}{X}$ and $\frac{X+Y}{Y} \leq \frac{Y+H}{Y}$,for some H $\ll_T$M .

Before we give our next result , we need the following proposition .

Proposition 2.2 [2] Let M be a module and let T,X and Y be submodules of M .

1- If X ≤ Y and Y $\ll_T$ M ,then X $\ll_T$ M .

2- If X $\ll_T$ M and Y $\ll_T$ M ,then X+Y $\ll_T$ M .

3- Let f: M→N be a homomorphism and X $\ll_T$ M ,then f(X) $\beta_{f(T)}^*$ f(Y) .

Lemma 2.3 The β_T^* relation is an equivalent relation .

Proof .

The reflexive and symmetric properties are clear .For transitively ,assume that X β_T^* Y and Y β_T^* Z .Then there is $H_1 \ll_T M$ and $H_2 \ll_T M$ such that $\frac{X+Y}{X} \leq \frac{X+H1}{X}$ and $\frac{X+Y}{Y} \leq \frac{Y+H1}{Y}$, $\frac{Y+Z}{Y} \leq \frac{Y+H2}{Y}$ and $\frac{Y+Z}{Z} \leq \frac{Z+H2}{Z}$.Hence $X+Y \leq X+H_1$, $X+Y \leq Y+H_1$ and $Y+Z \leq Y+H_2$,$Y+Z \leq Z+H_2$ .Now $X+Z \leq X+Y+Z \leq X+Y+H_2 \leq X + H_1 +H_2$ and $X+Z \leq X+Y+Z \leq Y+Z+H_1 \leq Z+H_1+H_2$ .Hence $\frac{X+Z}{X} \leq \frac{X+H1+H2}{X}$ and $\frac{X+Z}{Z} \leq \frac{Z+H1+H2}{Z}$ .Let $H=H_1+H_2$ .By proposition 2.2-2 , H $\ll_T$M .Therefore X β_T^* Z .Thus β_T^* is equivalent relation .

Note: Let M be a module ,then

1- X β_0^* Y ,for all X ,Y submodules of M .

2- Let X ,Y submodues of M such that X β_T^* Y .Then X $\ll_T$M if and only if Y $\ll_T$M .

Proposition 2.4 Let T be a submodule of M .Then submodules X , Y are β_T^* equivalent if and only if there exists H $\ll_T$M such that X+H=Y+H .

Proof .

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$\rightarrow$) Let X β_T^* Y, then there exists H $\ll_T$M such that $\frac{X+Y}{X} \le \frac{X+H}{X}$ and $\frac{X+Y}{Y} \le \frac{Y+H}{Y}$ .Now X+Y $\le$ X+Y +H $\le$ Y+H+H = Y+H and Y+H $\le$ X+Y+H $\le$ X+H+H = X+H .Thus X+H=Y+H .
$\leftarrow$) Assume there is H $\ll_T$M such that X+H=Y+H .Then X+Y $\le$ X+Y+H=X+X+H=X+H and X+Y $\le$ X +Y+H=Y+Y+H=Y+H .So $\frac{X+Y}{X} \le \frac{X+H}{X}$ and $\frac{X+Y}{Y} \le \frac{Y+H}{Y}$ .Thus X β_T^* Y.

Proposition 2.5 Let T be a submodule of M .Then submodules X,Y are β_T^* equivalent if and only if there exists H $\ll_T$M such that X+Y=X+H=Y+H .
Proof .
$\rightarrow$) Let X β_T^* Y .Then $\frac{X+Y}{X} \le \frac{X+H}{X}$ and $\frac{X+Y}{Y} \le \frac{Y+H}{Y}$, for some H $\ll_T$M .So X+Y $\le$ X+H and X+Y $\le$ Y+H .By modular law, X+Y = (X+Y)∩(X+H) = X+ ((X+Y)∩H) and X+Y = (X+Y)∩(Y+H) = Y+ ((X+Y) ∩H) .Let K= (X+Y)∩H $\le$ H .By proposition 2.2-1 ,K$\ll_T$M .Thus X+Y=X+H=Y+H .
$\leftarrow$) Clear .

Proposition 2.6 Let X , Y be submodues of a module M such that X $\le$ Y+H_1 and Y $\le$ X+H_2 ,for some T-small submodules H_1,H_2 of M ,then X β_T^* Y .
Proof .
Let X $\le$ Y+H_1 and Y $\le$ X+H_2 ,for some $H_1 \ll_T$M and $H_2 \ll_T$M .Then X+Y $\le$ X+X+ H_2 = X+ H_2 $\le$ X+H_1+ H_2 and X+Y $\le$ Y+Y+H_1 = Y+H_1 $\le$ Y+H_1+H_2 .Let H = H_1+H_2 .So $\frac{X+Y}{X} \le \frac{X+H}{X}$ and $\frac{X+Y}{Y} \le \frac{Y+H}{Y}$, for some H $\ll_T$M ,by proposition 2.2-2 .Thus X β_T^* Y .

Proposition 2.7 Let X_1 ,X_2 ,Y_1 ,Y_2 be submodules of a module M such that X_1 β_T^* Y_1 and X_2 β_T^* Y_2 ,then (X_1+X_2) β_T^* (Y_1+Y_2) .
Proof .
Assume that X_1 β_T^* Y_1 and X_2 β_T^* Y_2 .Then there exists $H_1 \ll_T$M and $H_2 \ll_T$M such that X_1+H_1=Y_1+H_1 and X_2+H_2 = Y_2+H_2 .Hence X_1+X_2+H_1+H_2 = Y_1+Y_2+H_1+H_2 .Let H = H_1+H_2 .Then H $\ll_T$M ,by proposition 2.2-2 .So X_1+X_2+H = Y_1+Y_2+H .Thus (X_1+X_2) β_T^* (Y_1+Y_2) .
Corollary 2.8 Let X ,Y and H be submodules of a module M such that H $\ll_T$M .Then X β_T^* Y if and only if X β_T^* (Y+H) .
Proposition 2.9 Let f: M $\rightarrow$N be an homomorphism and X ,Y be submodules of M such that X β_T^* Y ,then f(X) $\beta_{f(T)}^*$ f(Y) .
Proof .
Let X β_T^* Y , then there exists H $\ll_T$ M such that X+H = Y+H .Hence f(X)+ f(H) = f(X+H) = f(Y+H) = f(Y) + f(H) and f(H) $\ll_{f(T)}$ N ,by proposition 2.2-3 .Thus f(X) $\beta_{f(T)}^*$ f(Y) .

3- Submodules β_T^* equivalent to a direct summand .

In this section ,we develop the properties of submodules β_T^* equivalent to a direct summand .

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Definition 3.1 Let T be a submodule of a module M .We say that M is Goldie*-T-lifting ,G*-T-lifting, if for each submodule X of M ,there exists a direct summand D of M such that X β_T^* D.

Proposition 3.2 L et M be a module .Then the following statements are equivalent:
1- M is G*-M-lifting module.
2- M is G*-lifting module .
3- M is H-supplemented module .
Proof .
1→2) Let M be G*-M-lifting and let X be submodule of M .Then there exists a direct summand D of M such that $\frac{X+D}{X} \leq \frac{X+H}{X}$ and $\frac{X+D}{D} \leq \frac{D+H}{D}$, for some H$\ll_M$M .By [2,P.44] ,H $\ll$M .Now let π_X :M$\to \frac{M}{X}$ and
π_D:M$\to \frac{M}{D}$ be the natural epimorphisms .Then π_X(H) = $\frac{X+H}{X} \ll \frac{M}{X}$ and π_D(H) = $\frac{D+H}{D} \ll \frac{M}{D}$,By [2] ,so $\frac{X+D}{X} \ll \frac{M}{X}$ and $\frac{X+D}{D} \ll \frac{M}{D}$ .Hence X β^*D .Thus M is G*-lifting .
2→1) Let M be G*-lifting and let X be a submodule of M .By [5],then there exists a direct summand D of M and S $\ll$M such that X+D=D+H =X+H .By [2,P.44] ,S$\ll_T$M .By proposition (2.4) X β_M^* D .Thus M is G*-M -lifting .
2↔3) By [5,P.49] .

Proposition 3.3 Let T be a submodule of a module M .Consider the following statements :
1- For each submodule X of M, there exists a decomposition M=D+D`=X+D` and (X+D)∩D` $\ll_T$M .
2- For each submodule X of M, there exists a decomposition M=D+D`=X+D` and S $\ll_T$M such that X+D
=D⊕S=X+S .
3- For each submodule X of M, there exists a decomposition M=D+D`=X+D` such that X β_T^* D.
4- M is G*-T-lifting module.
Then **1→2→3→4** .If every T-small submodule of M is projective invariant, then **3→1**.
Proof .
1→2) Let X be a submodule of M .Then there exists a decomposition M=D+D`=X+D` and (X+D)∩D`$\ll_T$ M .By modular law X+D= (X+D)∩(D⊕D`) =D⊕((X+D)∩D`) , X+D= (X+D)∩(X+D`) =X+((X+D) ∩D`) .Let S= ((X+D)∩D`) .Thus X+D=D⊕S=X+S .
2→3) Clear by proposition (2.4) .
3→4) Clear by proposition (2.4) .
Now, assume that every T-small submodule of M is projective invariant and let X be a submodule of M. Then there exists a decomposition M=D+D`=X+D` such that X β_T^* D .By proposition (2.4) ,there exists S $\ll_T$M such that X+D=X+S=D+S .By modular law , X+D=

$(X+D)\cap(D\oplus D`) = D\oplus((X+D)\cap D`)$ .Let $P:M \to D`$ be the projection map .Then $P(S) = P(D+S) = P(X+D) = P((X+D)\cap D`) = (X+D)\cap D`$ .But $S \ll_T M$ and projective invariant ,then $P(S) = (X+D)\cap D` \leq S$ .By proposition 2.2-1 ,$(X+D)\cap D` \ll_T M$.

Proposition 3.4 Let T be a submodule of a module M .Consider the following statements:

1- For each submodule X of M ,there exists $\alpha\in End(M)$ such that $\alpha^2 = \alpha$,$M = X+(1-\alpha)(M)$ and $(1-\alpha)(X)$ $\ll_T M$.

2- For each submodule X of M , there exists a decomposition $M = D+D` = X+D`$ and $(X+D)\cap D` \ll_T M$.

3- M is G*-T-lifting module.

Then **1↔2→3 .**

Proof .

1→2) Let X be a submodule of M .Then there exists $\alpha\in End(M)$ such that $\alpha^2 = \alpha$, $M = X+ (1-\alpha)(M)$ and $(1-\alpha)(X) \ll_T M$ .Let $D = \alpha(M)$ and $D` = (1-\alpha)(M)$ .So $M = D\oplus D` = X+D`$ .Now ,$(X+D)\cap D` = (X+D)\cap(1-\alpha)(M)$ .Claim that $(X+D)\cap(1-\alpha)(M) = (1-\alpha)(X+D)$ .To show that let $a = (1-\alpha)(b) \in (X+D)\cap(1-\alpha)(M)$ .Since $(1-\alpha)^2 = 1-\alpha$,then $a = (1-\alpha)^2(b) = (1-\alpha)(a) \in (1-\alpha)(X+D)$ .Now ,let $a = (1-\alpha)(b) \in \alpha(X+D)$,then $a \in (1-\alpha)(M)$.$a = b-\alpha(b) \in X+D$ .Thus $a \in (X+D)\cap(1-\alpha)(M)$ .Now ,$(1-\alpha)(X) = (1-\alpha)(X+D) = (X+D)\cap(1-\alpha)(M) = (X+D)\cap D` \ll_T M$.

2→1) Let X be a submodule of M .There exists a decomposition $M = D+D` = X+D`$ and $(X+D)\cap D` \ll_T M$.

Let $\alpha: M \to D$ be the projection map .Clearly that $\alpha^2 = \alpha$ and $M = \alpha(M)\oplus(1-\alpha)(M) = X+(1-\alpha)(M)$ .Now ,$(1-\alpha)(X) = (1-\alpha)(X+D) = (X+D)\cap(1-\alpha)(M) = (X+D)\cap D` \ll_T M$.

2→3) Clear by proposition (3.3) .

Now we introduce the following two concepts.

Definition 3.5 Let T be a submodule of M .We say that M is T-lifting if for each submodule X of M ,there exists a direct summand D of M such that $D \leq X$ and $X\,\beta_T^*\,D$.

Clearly that by proposition 2.5 ,M is T-lifting if and only if for each submodule X of M ,then there exists a direct summand D and $H \ll_T M$ such that $X = D+H$.

By [6] and [2] ,M is M-lifting if and only if M is lifting .

A lifting module need not be T-lifting ,$\forall T \leq M$ .For example ,consider Z_4 as Z-module and let $T=\{\bar{0},\bar{2}\}$one can easily show that Z_4 is lifting but not T-lifting .

Clearly that indecomposable module M is not T-lifting ,$\forall 0\neq T \lneqq M$,For example ,Z as Z-module .

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Definition 3.6 Let T be a submodule of a module M .We say that M is T-H-supplemented module if for each submodule X of M ,there exists a direct summand D of M such that T $\subseteq$ X+ K if and only if T $\subseteq$ D+ K ,for every submodule K of M .
Clearly that M is M-H-supplemented if and only if M is H-supplemented .It is known that M is H-supplemented if and only if M is G*-lifting ,see [5,P.49] .

Proposition 3.7 L et M be a module .Then the following statements are equivalent:
1- M is T-H-supplemented module .
2- For each submodule X of M, there exists D $\leq\oplus$M such that for each A $\leq$ M with T $\subseteq$ X+D+A ,then T $\subseteq$X +A and T $\subseteq$ D+A .
3- For each X $\leq$ M , then there exists D $\leq\oplus$M such that $\frac{X+D}{X} \ll_{\frac{T+X}{X}} \frac{M}{X}$ and $\frac{X+D}{D} \ll_{\frac{T+D}{D}} \frac{M}{D}$.

Proof .
1$\rightarrow$2) Let X be a submodule of M .Then there exists D $\leq\oplus$M satisfies (1) .Now, let A be a submodule of M and T $\subseteq$X+D+A .By (1) , T $\subseteq$ X+ (X+A) = X+A and T $\subseteq$ D+ (D+A) = D+A .
2$\rightarrow$1) Let X be a submodule of M .Then there exists D $\leq\oplus$M satisfies (2) .Let K be a submodule of M such that T $\subseteq$ X+K .Then T $\subseteq$ X+D+K .By (2) ,T $\subseteq$ D+K .Let H be a submodule such that T $\subseteq$ D+H .Then T $\subseteq$ X+D+H .By (2) , T $\subseteq$ X+H .Thus M is T-H-supplemented module .
2$\rightarrow$3) Let X be a submodule of M .Then there exists D $\leq\oplus$M satisfies (2) .To show $\frac{X+D}{X} \ll_{\frac{T+X}{X}} \frac{M}{X}$,Let $\frac{T+X}{X} \subseteq \frac{X+D}{X} + \frac{A}{X}$ .Then T $\subseteq$ X+D+A .By (2) ,T $\subseteq$ X+A = A .Thus $\frac{T+X}{X} \subseteq \frac{A}{X}$ .By the same way $\frac{X+D}{D} \ll_{\frac{T+D}{D}} \frac{M}{D}$.
3$\rightarrow$2) Let X $\leq$ M,then there exists D$\leq\oplus$M such that $\frac{X+D}{X} \ll_{\frac{T+X}{X}} \frac{M}{X}$ and $\frac{X+D}{D} \ll_{\frac{T+D}{D}} \frac{M}{D}$ .Let A be a submodule of M such that T $\subseteq$ X+D+A .Now $\frac{T+X}{X} \subseteq \frac{X+D+A}{X} = \frac{X+D}{X} + \frac{A+X}{X}$ .Since $\frac{X+D}{X} \ll_{\frac{T+X}{X}} \frac{M}{X}$,then $\frac{T+X}{X} \subseteq \frac{A+X}{X}$ and hence T $\subseteq$ X+A .By the same way T $\subseteq$ D+A .

Proposition 3.8 Let T be a submodule of a module M .If M is G*-T-lifting modue ,then M is T-H-supplemented module .
Proof.
Let X be a submodule of M .there exists a direct summand D of M such that $\frac{X+D}{X} \leq \frac{X+H}{X}$ and $\frac{X+D}{D} \leq \frac{D+H}{D}$,for some H $\ll_T$M .Let $\pi_X : M \rightarrow \frac{M}{X}$ and $\pi_D : M \rightarrow \frac{M}{D}$ be the natural epimorphisms .Since H $\ll_T$ M .By proposition 2.2-3 ,$\pi_X(H) = \frac{X+H}{X} \ll_{\frac{T+X}{X}} \frac{M}{X}$ and $\pi_D(H) = \frac{D+H}{D} \ll_{\frac{T+D}{D}} \frac{M}{D}$ .Thus M is H-supplemented module .

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